Negative beta transformations

Lingmin Liao
Joint with Wolfgang Steiner (University Paris 7)

Université Paris-Est Créteil (Paris 12)

Universität Bremen, Bremen
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Outline

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The $(-\beta)$-transformation and $(-\beta)$-shift
0. The $\beta$-transformation and the $(-\beta)$-transformation

**Figure:** $\beta$-transformation (left) and $(-\beta)$-transformation (right), $\beta = \frac{1 + \sqrt{5}}{2}$.
I. The $\beta$-transformation

Rényi’s $\beta$-transformation $T_\beta : [0, 1) \to [0, 1)$ is defined by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor.$$

Let

$$d_{\beta,1}(x) = \lfloor \beta x \rfloor, \quad d_{\beta,n}(x) = d_{\beta,1}(T_\beta^{n-1}(x)) \quad \text{for } n \geq 1.$$

Then

$$x = \frac{\lfloor \beta x \rfloor}{\beta} + \frac{T_\beta x}{\beta} = \frac{d_{\beta,1}}{\beta} + \frac{d_{\beta,2}}{\beta^2} + \frac{d_{\beta,3}}{\beta^3} + \ldots.$$

Sequence $d_\beta(x) = d_{\beta,1}(x)d_{\beta,2}(x)\cdots \longrightarrow \beta\text{-expansion of } x$.

**Example**: $\beta = \frac{1+\sqrt{5}}{2}$,

$$1 = \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{0}{\beta^3} + \frac{0}{\beta^4} + \cdots.$$

$\rightarrow$ expansion of $1 = 11\overline{0} = 11000\ldots$. 
II. The \((-\beta)\)-transformation

Define \(T_{-\beta} : (0, 1] \rightarrow (0, 1]\) by

\[
T_{-\beta}(x) := -\beta x + \lfloor \beta x \rfloor + 1.
\]

Let

\[
d_{-\beta,1}(x) = \lfloor \beta x \rfloor + 1, \quad d_{-\beta,n}(x) = d_{-\beta,1}(T_{-\beta}^{n-1}(x)) \quad \text{for } n \geq 1.
\]

Then

\[
x = \frac{-d_{-\beta,1}(x)}{-\beta} + \frac{T_{-\beta}(x)}{-\beta} = \frac{-d_{-\beta,1}(x)}{-\beta} + \frac{-d_{-\beta,2}(x)}{(-\beta)^2} + \frac{-d_{-\beta,3}(x)}{(-\beta)^3} + \cdots.
\]

Sequence \(d_{-\beta}(x) = d_{-\beta,1}(x)d_{-\beta,2}(x) \cdots \rightarrow \text{\((-\beta)\)-expansion of } x.\)

**Example**: \(\beta = \frac{1+\sqrt{5}}{2},\)

\[
1 = \frac{-2}{-\beta} + \frac{-1}{(-\beta)^2} + \frac{-1}{(-\beta)^3} + \frac{-1}{(-\beta)^4} + \cdots.
\]

\(\rightarrow\) expansion of \(1 = 2\bar{1} = 2111\ldots\)
III. Remarks about the definition

- **Ito and Sadahiro 2009** : On the interval $I_\beta = [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$:

  $$T_{IS} : x \mapsto -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor.$$  

  We have the conjugacy:

  $$\begin{align*}
  (0, 1] & \xrightarrow{T_{-\beta}} (0, 1] \\
  \frac{1}{\beta+1} - x & \xrightarrow{T_{IS}} \frac{1}{\beta+1} - x \\
  I_\beta & \xrightarrow{T_{IS}} I_\beta
  \end{align*}$$

  So all results can be translated to our case.

- **Our definition is one case of generalized $\beta$-transformations studied by Góra 2007 and Faller 2008 (Ph.D Thesis ).**
IV. Parry’s admissible sequence and $\beta$-shift

A sequence $a_1a_2\cdots$ is said **admissible** if $\exists x \in (0, 1], d_\beta(x) = a_1a_2\cdots$.

**Lexicographical order**: $a_1a_2\cdots \prec b_1b_2\cdots$ if and only if

$$\exists k \geq 1, \quad a_i = b_i \text{ for } i < k \quad \text{and} \quad a_k < b_k.$$

Denote $a_1a_2\cdots \preceq b_1b_2\cdots$, if $a_1a_2\cdots \prec b_1b_2\cdots$ or $a_1a_2\cdots = b_1b_2\cdots$.

The **$\beta$-shift** $S_\beta$ on the alphabet $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ is the closure of the set of admissible sequences.

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**Theorem (Parry 1960)**

A sequence $a_1a_2\cdots$ is admissible if and only if for each $n \geq 1$

$$a_na_{n+1}\cdots \prec d_\beta^*(1).$$

A sequence $a_1a_2\cdots$ is in $S_{-\beta}$ if and only if for each $n \geq 1$

$$a_na_{n+1}\cdots \preceq d_\beta^*(1).$$

Here $d_\beta^*(1) := \lim_{x \to 1^-} d_\beta(x)$. 
V. Admissible sequence and \((-\beta)\)-shift

A sequence \(a_1a_2\cdots\) is said **admissible** if \(\exists x \in (0, 1], d_{-\beta}(x) = a_1a_2\cdots\).

**Alternate order**: \(a_1a_2\cdots \prec b_1b_2\cdots\) if and only if

\[
\exists k \geq 1, \quad a_i = b_i \text{ for } i < k \quad \text{and} \quad (-1)^k(b_k - a_k) < 0.
\]

Denote \(a_1a_2\cdots \preceq b_1b_2\cdots\), if \(a_1a_2\cdots \prec b_1b_2\cdots\) or \(a_1a_2\cdots = b_1b_2\cdots\).

The \((-\beta)\)-shift \(S_{-\beta}\) on the alphabet \(\{1, \ldots, \lfloor \beta \rfloor + 1\}\) is the closure of the set of admissible sequences.

**Theorem (Ito-Sadahiro 2009)**

A sequence \(a_1a_2\cdots\) is admissible if and only if for each \(n \geq 1\)

\[
d^*_{-\beta}(0) < a_na_{n+1}\cdots \preceq d_{-\beta}(1).
\]

A sequence \(a_1a_2\cdots\) is in \(S_{-\beta}\) if and only if for each \(n \geq 1\)

\[
d^*_{-\beta}(0) \preceq a_na_{n+1}\cdots \preceq d_{-\beta}(1).
\]

Here \(d^*_{-\beta}(0) := \lim_{x \to 0^+} d_{-\beta}(x)\).
We have

\[ d_{-\beta}^*(0) = \begin{cases} 
1b_1b_2 \cdots b_{q-1}(b_q - 1), & \text{if } d_{-\beta}(1) = b_1 \cdots b_{q-1}b_q \text{ for some odd } q \\
1d_{-\beta}(1) & \text{otherwise.}
\end{cases} \]

The sequence \( d_{-\beta}^*(0) = \lim_{x \to 0^+} d_{-\beta}(x) \) is the orbit sequence of 0 under \( \hat{T}_{-\beta} : [0, 1] \to [0, 1] \):

Here we have modified \( T_{-\beta} \) by : \( \hat{T}_{-\beta}(0) = 1 \) and \( \hat{T}_{-\beta}(\frac{k}{\beta}) = 0 \).
VI. Admissible sequence and \((-\beta)\)-shift (continued)

Two important subshifts:

- subshift of finite type: finite words are forbidden.
- sofic (Hebrew word for "finite"): factor of subshift of finite type.

We have:

- The \(\beta\)-shift is of finite type \(\iff d_\beta(1)\) is finite.
- The \(\beta\)-shift is sofic \(\iff d_\beta(1)\) is eventually periodic.

**Theorem (Ito-Sadahiro 2009)**

*The \((-\beta)\)-shift is sofic if and only if \(d_{-\beta}(1)\) is eventually periodic.*

**Theorem (Frougny-Lai 2009)**

*The \((-\beta)\)-shift is of finite type if and only if \(d_{-\beta}(1)\) is purely periodic.*
VII. Sequences being an \((-\beta)\)-expansion of 1

Parry 1960: A sequence \(a_1a_2a_3\cdots\) is the \(\beta\)-expansion of 1 for some \(\beta > 1\) if and only if

\[ a_ka_{k+1}a_{k+2}\cdots \preceq_{\text{lex}} a_1a_2a_3\cdots \quad \forall k \geq 2. \]

Theorem (Steiner, Acta Math Hung 2013)

A sequence \(a_1a_2a_3\cdots\) is the \((-\beta)\)-expansion of 1 for some \(\beta > 1\) if and only if

1. \(a_ka_{k+1}a_{k+2}\cdots \preceq_{\text{alt}} a_1a_2a_3\cdots \quad \forall k \geq 2.\)
2. \(a_1a_2a_3\cdots \succeq_{\text{alt}} u_1u_2\cdots := 2112221121121122211\cdots.\)
3. For all \(k\), with \(\overline{a_1\cdots a_k} \succeq u_1u_2\cdots\), we have

\[ a_1a_2a_3\cdots \notin \{a_1\cdots a_k, a_1\cdots a_k-1(a_k - 1)1\}^\omega \setminus \{\overline{a_1\cdots a_k}\}. \]
4. For all \(k\), with \(\overline{a_1\cdots a_{k-1}(a_k + 1)} \succeq u_1u_2\cdots\), we have

\[ a_1a_2a_3\cdots \notin \{a_1\cdots a_k1, a_1\cdots a_k-1(a_k + 1)\}^\omega. \]
VIII. Isomorphisms between $\beta$ and $-\beta$

Observations:

- Topologically entropy are the same: $\log \beta$.
- Cannot be topologically conjugate to each other since the number of fixed points are different.

Multinacci number $\beta_n$: the unique root in $(1, 2)$ of the polynomial

$$x^n - x^{n-1} - \cdots - x - 1.$$

Theorem (Kalle, ETDS to appear)

For the multinacci number $\beta_n$, the $\beta_n$-transformation and the $(-\beta_n)$-transformation are measurably isomorphic.

For others, they are not measurably isomorphic.
IX. The greedy and lazy $\beta$-representations

Observations: we have many ways (usually uncountable) to write a real number in base $\beta$.

Question: Which expansion is the biggest in the lexicographical order, and which is the smallest? How to obtain them?

Answer: Rényi’s $T_\beta$ gives the ”biggest” one. See many works by Dajani and Kraaikamp.

The following is an example for golden ratio.

Figure 1: The greedy and lazy transformations $T_G$ and $T_L$ for $\beta = +\phi$. 
X. The greedy and lazy $(-\beta)$-representations

Observations: we have many ways (usually uncountable) to write a real number in base $(-\beta)$.

Question: Which expansion is the biggest in the alternate order, and which is the smallest? How to obtain them?

Hejda, Masáková, Pelantová (Kybernetika 2013):
- Ito and Sandahiro’s transformation and $T_{-\beta}$ can never give the ”biggest” one, nor the ”smallest” one.
- The greedy and lazy $(-\beta)$-representations can be obtained by using $T_G = T_1 \circ T_0$, and $T_L = T_0 \circ T_1$.

![Graphs of the transformations $T_0, T_1 : [-1, \frac{1}{\phi}] \mapsto [-1, \frac{1}{\phi}]$.](image-url)
XI. \( \beta \)-integers

\( \beta \)-integers \( \mathbb{Z}_\beta \) :

\[
\pm \sum_{k=0}^{n-1} a_k \beta^k, \text{ s.t. } 0 \leq \sum_{k=0}^{m-1} a_k \beta^k < \beta^m \forall 1 \leq m \leq n,
\]

i.e., \( \sum_{k=0}^{n-1} a_k \beta^k \) is a greedy \( \beta \)-expansion. We have

\[
\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+), \quad \mathbb{Z}_\beta^+ = \cup_{n \geq 0} \beta^n T^{-n}_\beta(0).
\]

Thurston, 1989: the set \( \Delta_\beta \) of distances between consecutive elements of \( \mathbb{Z}_\beta \) is

\[
\Delta_\beta = \{ T^m_\beta (1-) : n \geq 0 \}.
\]

Fabre, 1995: if \( \beta \) is a Parry number, then \( \Delta_\beta \) is finite and the sequence of distances in \( \mathbb{Z}_\beta^+ \) is coded by the fixed point of a substitution.
XII. \((-\beta)\)-integers

\((-\beta)\)-integers \(\mathbb{Z}_{-\beta}\):

\[
\sum_{k=0}^{n-1} a_k (-\beta)^k, \text{ s.t. } \frac{-\beta}{\beta+1} \leq \sum_{k=0}^{m-1} a_k (-\beta)^{k-m} < \frac{1}{\beta+1} \forall 1 \leq m \leq n,
\]

i.e., \(\sum_{k=0}^{n-1} a_k (-\beta)^k\) is an Ito-Sadahiro \((-\beta)\)-expansion.

Ambrož-Dombek-Masáková-Pelantová, Steiner, 2012:

- if \(\beta < (1 + \sqrt{5})/2\), \(\mathbb{Z}_{-\beta} = \{0\}\) and if \(\beta \geq (1 + \sqrt{5})/2\),

\[
\mathbb{Z}_{-\beta} \cap (-\beta)^n [-\beta, 1] = \{(-\beta)^n, (-\beta)^{n+1}\} \cup (-\beta)^{n+2} (T_{-\beta}^{-n-2}(0) \cap (\frac{-1}{\beta}, \frac{1}{\beta^2})�).
\]

- discussion about the set \(\Delta_{-\beta}\) of distances between consecutive elements of \(\mathbb{Z}_{-\beta}\) (complicated).

- if \(\beta\) is a Yrrap number, then \(\Delta_{-\beta}\) is finite and the structure of \(\mathbb{Z}_{-\beta}\) can be described by the fixed point of an anti-morphism.
XIII. \((-\beta)\)-integers (continued)

The set of \(\beta\)-integers \(\mathbb{Z}_\beta\) is

- **relatively dense** (there exists \(R > 0\) such that each interval of length \(R\) contains at least one point of \(\mathbb{Z}_\beta\))

- **uniformly discrete** (there exists \(r > 0\) such that each interval of length \(r\) contains at most one point of \(\mathbb{Z}_\beta\) if and only if \(0\) is not an accumulation point of \(\{T^n_\beta(1)|n \geq 0\}\).

For the set of \((-\beta)\)-integers \(\mathbb{Z}_{-\beta}\):

**Steiner, 2011**:

- \(\mathbb{Z}_{-\beta}\) is not necessarily relatively dense.

- if \(0\) is not an accumulation point of \(\{T^{2n-1}_{-\beta}(\frac{-\beta}{\beta+1}) > 0|n \geq 1\}\), then the set \(\mathbb{Z}_{-\beta}\) is uniformly discrete. (No sufficient and necessary condition is known.)
Dynamical properties of 
\((-\beta)\)-transformation
I. Some notions of dynamical systems

Suppose $T : X \to X$ be a dynamical system.

- **locally eventually onto**: if for every nonempty open subset $U \subset X$, there exists a positive integer $n_0$ such that for every $f^{n_0}(U) = X$.

- **exactness**: $T$ acting on $(X, \mathcal{B}, \mu)$ is called exact if

  $$\bigcap_{n=0}^{\infty} T^{-n} \mathcal{B} = \{X, \emptyset\}$$

  or equivalently, for any positive measure set $A$ with $T^n(A) \in \mathcal{B}$ ($n \geq 0$),

  $$\mu(T^n(A)) \to 1 \ (n \to \infty).$$

- **maximal entropy measure**: the measure attaining the maximum of

  $$\sup\{h_\mu : \mu \text{ invariant}\}.$$

- **intrinsic ergodicity**: the maximal entropy measure is unique.
II. General piecewise monotone transformation

$T : [0, 1] \rightarrow [0, 1]$.

- finite partition of $[0, 1] : P = \{P_1, \cdots, P_N\}$.
- on each $P_i$, $T$ is monotonic, Lipschitz continuous and $|T'| \geq \rho > 1$.

**Lasota-Yorke 1974**: There is an invariant measure $d\mu = hd\lambda$, where $d\lambda$ is the Lebesgue measure and $h$ is a density of bounded variation.

**Keller 1978**: The set $\{h \neq 0\}$ is a **finite** union of intervals.

**Wagner 1979**: We can decompose $[0, 1] = \bigcup_{i=1}^{s} A_i \cup B$, such that

- on each $A_i$ there is an invariant measure which is equivalent to the Lebesgue measure restricted to $A_i$
- each $A_i$ can be decomposed as $A_i = \bigcup_{j=1}^{m_i} A_{ij}$ and $T^{m_i}$ is **exact** on each $A_{ij}$.
- the set $B$ satisfies $T^{-1}B \subset B$ and $\lim_{n \to \infty} \lambda(T^{-n}B) = 0$.

**Hofbauer 1981**: The number of maximal entropy measures is **finite**. If $T$ is topological transitive, then it is **intrinsic ergodic**.
III. Dynamical properties : $\beta$ and $(-\beta)$

**Rohklin 1960** : The transformation $T_\beta$ is exact for all $\beta > 1$.

**Smorodinsky 1973** : The transformation $T_\beta$ is week Bernoulli for all $\beta > 1$.

**Hofbauer 1978** : The transformation $T_\beta$ admits a unique maximal measure for all $\beta > 1$.

**Góra 2007** : for $\beta > \gamma_1 = 1.3247...$ (the smallest Pisot number), the transformation $T_{-\beta}$ is exact and he conjectured that this would hold for all $\beta > 1$.

**Faller 2008** : $\beta > \sqrt[3]{2}$, $T_{-\beta}$ admits a unique maximal measure.

**Question** : What happens for the small beta’s?
IV. Invariant measures

**Gel’fond 1959, Parry 1960**: There is a unique (with a constant difference) measure $h_\beta(x)dx$ which is invariant under $T_\beta$, where

$$h_\beta(x) = \sum_{n \geq 0, T_\beta^n(1) > x} \frac{1}{\beta^n}.$$

It is equivalent to the Lebesgue measure:

$$(\beta - 1)/\beta \leq h_\beta(x) \leq \beta/((\beta - 1)).$$

**Ito-Sadahiro 2009**: Let $h_{-\beta}$ be a real-valued function defined on $(0, 1]$ by

$$h_{-\beta}(x) = \sum_{n \geq 0, T_{-\beta}^n(1) \geq x} \frac{1}{(-\beta)^n}.$$

Then the measure $h_{-\beta}(x)dx$ is an invariant measure of $T_{-\beta}$.

**Remark**: The density may be zero on some intervals. So the invariant measure is not equivalent to the Lebesgue measure. (Different to the $\beta$ case).
V. An example

Take $\gamma_1 > 1$ be the root of $x^3 = x + 1$. We can find that $T^k(1) = T^3(1)$ and

$$0 < T^2(1) < T^1(1) < T^3(1) < 1.$$  

By the definition of $h_{-\beta}$, we have

on $(0, T^2(1))$, 

$$h_{-\beta}(x) = \sum_{n \geq 0} \frac{1}{(-\beta)^n} = \frac{1}{1 + \frac{1}{\beta}},$$

on $(T^2(1), T^1(1))$, 

$$h_{-\beta}(x) = (\sum_{n \geq 0} \frac{1}{(-\beta)^n}) - \frac{1}{(-\beta)^2} = \frac{1}{1 + \frac{1}{\beta}} - \frac{1}{(-\beta)^2} = 0,$$

on $(T^1(1), T^3(1))$, we have $h_{-\beta}(x) = 0 - 1/(-\beta)^3 = 1/\beta^3$, 

and finally on $(T^3(1), 1)$, $h_{-\beta}(x) = 1$.

→ We have one interval on which the density is zero.
VI. How many gaps?

A question:
For a given $\beta$, how many intervals (gaps) on which the density $h_{-\beta}$ equals to 0?

When $\beta$ decreases, the numbers should be like

$0, 1, 2, 5, 10, 21,$

What is the next?
VII. Our results-Notations
For each \( n \geq 0 \), let \( \gamma_n \) be the positive real number defined by

\[
\gamma_{n+1} = \gamma_n + 1, \quad \text{with} \quad g_n = \lfloor \frac{2^{n+2}}{3} \rfloor.
\]

Then

\[
2 > \gamma_0 > \gamma_1 > \gamma_2 > \cdots > 1.
\]

Note that \( \gamma_0 \) is the golden ratio and that \( \gamma_1 \) is the smallest Pisot number.

For each \( n \geq 0 \) and \( 1 < \beta < \gamma_n \), set

\[
G_n(\beta) = \left\{ G_{m,k}(\beta) \mid 0 \leq m \leq n, \ 0 \leq k < \frac{2^{m+1} + (-1)^m}{3} \right\},
\]

with open intervals

\[
G_{m,k}(\beta) = \begin{cases} 
(T_{-\beta}^{2^m+1+k}(1), \ T_{-\beta}^{(2^m+2-(-1)^m)/3+k}(1)) & \text{if } k \text{ is even}, \\
(T_{-\beta}^{(2^m+2-(-1)^m)/3+k}(1), \ T_{-\beta}^{2^m+1+k}(1)) & \text{if } k \text{ is odd}.
\end{cases}
\]
VIII. Our results-Theorems
We call an interval a gap if the density of the invariant measure is zero on it.

Theorem (L-Steiner, ETDS 2012)

If $\beta \geq \gamma_0$, then there is no gap. If $\gamma_{n+1} \leq \beta < \gamma_n$, $n \geq 0$, then the set of gaps is $G_n(\beta)$ which consists of $g_n = \left\lfloor \frac{2^{n+2}}{3} \right\rfloor$ disjoint non-empty intervals.

Define

$$G(\beta) = \begin{cases} \emptyset & \text{if } \beta \geq \gamma_0, \\ \bigcup_{I \in G_n(\beta)} I & \text{if } \gamma_{n+1} \leq \beta < \gamma_n, n \geq 0. \end{cases}$$

Theorem (L-Steiner, ETDS 2012)

The transformation $T_{-\beta}$ is locally eventually onto on $(0,1] \setminus G(\beta)$,

$$T_{-\beta}^{-1}(G(\beta)) \subset G(\beta) \quad \text{and} \quad \lim_{n \to \infty} \lambda(T_{-\beta}^{-n}(G(\beta))) = 0.$$
IX. Our results - Theorems (continued)

Define a morphism on the symbolic space $\{1, 2\}^\mathbb{N}$ by

$$\varphi : 1 \mapsto 2, \quad 2 \mapsto 211.$$

**Theorem (L-Steiner, ETDS 2012)**

For every $n \geq 0$, we have $d_{-\gamma_n}(1) = \varphi^n(2 1^\omega)$. Hence

$$\lim_{\beta \to 1} d_{-\beta}(1) = \lim_{n \to \infty} \varphi^n(2) = 21122211211211122 \cdots .$$

**Remarks**:

1. Thue-Morse sequence: $0110\ 1001\ 1001\ldots \rightarrow 0\ 110\ 1001\ 1001\ldots$
   Then count the numbers of consecutive ones and zeros:

   $11\ 0\ 1\ 00\ 1\ 1001\ldots$

   $2\ 1\ 1\ 2$

2. $|\varphi^m(2)| = |\varphi^{m+1}(1)| = g_n + \frac{1-(-1)^n}{2}$ and $|\varphi^m(21)| = 2^{m+1}$. 
X. Our results-Corollaries

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ is exact.

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ has a unique maximal measure, and hence is intrinsic ergodic.
XI. Our results-Proofs

For every word $a_1 \cdots a_n \in \{1, 2\}^n$, $n \geq 0$, define the polynomial

$$P_{a_1 \cdots a_n} = (-X)^n + \sum_{k=1}^{n} a_k (-X)^{n-k} \in \mathbb{Z}[X]$$

Lemma

For $1 \leq m < n$, we have

$$P_{a_1 \cdots a_n} = (-X)^{n-m} (P_{a_1 \cdots a_m} - 1) + P_{a_{m+1} \cdots a_n}.$$

For $n \geq 0$ we have the identities:

- $X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}}$

- $1 - P_{\varphi^n(1)} = X^{\frac{1+(-1)^n}{2}} \prod_{k=0}^{n-1} (X|\varphi^k(1)| - 1)$

- $P_{\varphi^n(21)} - P_{\varphi^n(2)} = (X^{g_n+1} - X - 1) \prod_{k=0}^{n-1} (X|\varphi^k(1)| - 1)$
XII. Our results-Proofs-continued

Let $1 < \beta < \gamma_n$, $n \geq 0$. Then the elements of $\mathcal{F}_n(\beta)$ and $\mathcal{G}_n(\beta)$ are intervals of positive length which form a partition of $(0, 1]$. Moreover, we have

1. $d_{-\beta}(1)$ starts with $\varphi^{n+1}(2)$, $T|\varphi^{n+1}(2)|(1) \in F_{n,0}$,
2. $1/\beta$ is an interior point of $F_{n,g_n-1}$,
3. $F_{n,k} = T^k(F_{n,0})$ for all $0 \leq k < g_n$,
4. $T^{g_n}(F_{n,0}) = F_{n,g_n} \cup F_{n,0}$, $T(F_{n,g_n}) = F_{n+1,0}$, if $n$ is even,
5. $T^{g_n}(F_{n,0}) = F_{n,g_n} \cup F_{n+1,0}$, $T(F_{n,g_n}) = F_{n,0}$, if $n$ is odd,
XIII. Hofbauer’s two parameter family

Figure: Piecewise linear transformation with two different negative slopes.

Theorem (Hofbauer 2012)

We can explicitly construct the non-wandering set which is a union of periodic orbits and some closed intervals. The number of the intervals are determined.
Yrrap number VS Parry number
I. Definitions

Extend the definition of $T_\beta$ to 1 by $T_\beta(1) := \beta - \lfloor \beta \rfloor$

- **Parry number** ($\beta$-number): number $\beta > 1$ such that the orbit of 1 under $T_\beta$ is eventually periodic.
- **Yrrap number** ($(-\beta)$-number): number $\beta > 1$ such that the orbit of 1 under $T_{-\beta}$ is eventually periodic.
- **Pisot number**: number $\beta > 1$, with all its algebraic conjugates $< 1$ in modulus.
- **Perron number**: number $\beta > 1$, with all its algebraic conjugates $< \beta$ in modulus.
II. Results

**Bertrand 1977, Schmidt 1980**: All Pisot numbers are Parry numbers.

**Frougny-Lai 2009**: All Pisot numbers are Yrrap numbers.

**Lind 1984, Denker-Grillenberger-Sigmund 1976**: All Parry numbers are Perron numbers.

**Masáková-Pelantová 2011**: All Yrrap numbers with modului $\geq 2$ are Perron numbers.

**Theorem (L-Steiner, ETDS 2012)**

*All Yrrap numbers are Perron numbers.*
III. Results-Continue

Lemma

Let \( \beta > 1 \) such that \( \beta^4 = \beta + 1 \), i.e., \( \beta \approx 1.2207 \). Then \( T^{10}_{-\beta}(1) = T^5_{-\beta}(1) \), and \( (T^n_{-\beta}(1))_{n \geq 0} \) is aperiodic.

Lemma

Let \( \beta > 1 \) such that \( \beta^7 = \beta^6 + 1 \), i.e., \( \beta \approx 1.2254 \). Then \( T^7_{-\beta}(1) = 0 \), and \( (T^n_{-\beta}(1))_{n \geq 0} \) is aperiodic.

Theorem (L-Steiner, ETDS 2012)

The set \((-\beta)\)-numbers and the set of \(\beta\)-numbers do not include each other.
Questions
I. About the dynamics

- Characterization of the $\beta$ such that the corresponding $(-\beta)$-shift satisfies the specification property:
  there exists $k \in \mathbb{N}$ such that for all admissible $v, w$, there exists $u$ with length less than $k$ and $vuw$ is also admissible.

- Characterization of the $\beta$ such that the corresponding $(-\beta)$-shift is synchronizing:
  containing a synchronizing word $u$, i.e., for all $v, w$, if $vu, wu$ are admissible, then

  $\text{vus is admissible} \Leftrightarrow \text{wus is admissible}$

- Chaotic properties.

- Natural extension. Are all $(-\beta)$-transformations are weak Bernoulli?
II. Classification and size


- Class C1. simple Parry numbers \( (S_\beta \) is a subshift of finite type) \( \rightarrow \) dense.
- Class C2. Parry numbers \( (S_\beta \) is a is sofic.) \( \rightarrow \) at most countable.
- Class C3. \( (S_\beta \) satisfies the specification property) \( \rightarrow \) Lebesgue measure 0, Hausdorff dimension 1.
- Class C4. \( (S_\beta \) is synchronizing) \( \rightarrow \) Lebesgue measure 0, Hausdorff dimension 1.
- Class C5. (the rest) \( \rightarrow \) Lebesgue measure 1.

**Question**: What is about the \(-\beta\) case?
III. Univoque set and size

Rényi’s $\beta$ case:

Let $J_\beta := [0, (\lceil \beta \rceil - 1)/(\beta - 1)]$. We are interested in the following set:

$$U := \{(x, \beta) : \beta > 1, x \in J_\beta, x \text{ has exactly one expansion in base } \beta\},$$

and the one dimensional sections:

$$U_\beta := \{x \in J_\beta : (x, \beta) \in U\}, \quad U := \{\beta > 1 : (1, \beta) \in U\}.$$

- $U : \text{Leb} = 0, \text{HD} = 2$ (de Vries-Komornik 2010).
- $U_\beta :$ (Glendinning-Sidorov 2001)
  - $1 < \beta \leq (1 + \sqrt{5})/2$ : two elements;
  - $(1 + \sqrt{5})/2 < \beta < \beta_{KL}$ : countably infinite;
  - $\beta_{KL} < \beta \leq 2$ : positive Hausdorff dimension.

Here $\beta_{KL} \approx 1.787$ is the Komornik-Loreti constant.

- $U :$ continuum many (Erdös-Horváth-Joó 1991),
  $$\text{Leb} = 0, \text{HD} = 1$$ (Daróczy-Kátai 1995).
IV. Schmidt conjecture

Salem Number: algebraic integer number $\beta > 1$, whose conjugates all have modulus $\leq 1$ and at least one $= 1$.

Denote $\text{Per}(\beta)$, $\text{Per}(-\beta)$ the sets of eventually periodic points for $T_\beta$ and $T_{-\beta}$ respectively.

Schmidt 1980: If $\mathbb{Q} \cap [0, 1) \subset \text{Per}(\beta)$, then $\beta$ is either a Pisot number or a Salem number.

Masáková-Pelantová 2010: If $\mathbb{Q} \cap (0, 1] \subset \text{Per}(-\beta)$, then $\beta$ is either a Pisot number or a Salem number.

Conversely,

Bertrand 1977: If $\beta$ is a Pisot number, then $\mathbb{Q} \cap (0, 1] \subset \text{Per}(\beta)$.

Frougny-Lai 2009: If $\beta$ is a Pisot number, then $\mathbb{Q} \cap (0, 1] \subset \text{Per}(-\beta)$.

Schmidt conjecture, 1980

If $\beta$ is a Salem number, then $\mathbb{Q} \cap (0, 1] \subset \text{Per}(\beta)$.

Question: Schmidt conjecture for $-\beta$ case?
V. Schmidt conjecture-progress (β case)

**Fact**: The degree of a Salem number is even and ≥ 4.

**Boyd 1989**: If $\beta$ is a Salem number of degree 4, then the orbit of 1 under $T_\beta$ is eventually periodic.

**Boyd 1996**: Some examples of Salem numbers of degree 6.

“There are also some very large orbits which have been shown to be finite: an example is given for which the preperiod length is 39420662 and the period length is 93218808”.